

Dimension bounds for invariant measures of bi-Lipschitz iterated function systems

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Abstract

We study probabilistic iterated function systems (IFS), consisting of a finite or infinite number of average-contracting bi-Lipschitz maps on \mathbb{R}^d . If our strong open set condition is also satisfied, we show that both upper and lower bounds for the Hausdorff and packing dimensions of the invariant measure can be found. Both bounds take on the familiar form of ratio of entropy to the Lyapunov exponent. Proving these bounds in this setting requires methods which are quite different from the standard methods used for average-contracting IFS's.

1 Introduction

When studying the dimension of measures for probabilistic iterated function systems, there are two common approaches. One method is to examine small balls centered at typical points for the Markov chain, and estimate the logarithmic densities (2.7)-(2.10). Since this method requires some intimate understanding of the geometry of the system, it has proven to be effective primarily in the simple case of similitudes satisfying the so-called open set condition, which limits the overlap of the maps. In this scenario, the exact dimensional value of the invariant measure μ has been determined in eg. [1] (the finite case) and later generalized in [6] (the infinite case).

Another common approach is to assume very little of the maps and only search for an upper bound s for the dimension of μ , by explicitly constructing a set of full μ -measure whose s -dimensional Hausdorff measure is zero. Usually only average contractivity (a notion introduced in [3]) is assumed. This approach has been used in eg. [9], [5] and [4]. While this method is easier to apply to a wider class of maps, the drawbacks are that it does not give any lower bound for the dimension, nor does it shed any light on the packing dimension of μ .

The aim of this paper is to obtain results for a special class of IFS by merging these two approaches in a way, primarily by extending techniques used for analyzing strictly contracting IFS's. We will focus on the case where the maps

are bi-Lipschitz. By assuming average contractivity, the support of μ is not necessarily bounded and many of the initial assumptions in the case of similitudes fail. We will extend some ideas from the second approach to show that we can still find lower and upper bounds for both the Hausdorff and packing dimensions of μ .

The motivation for this paper rises from the fact that lower bounds of the dimension of μ are not commonly investigated. One example is [8], where μ however is required to have bounded support. Here we will prove the intuitive result that the lower Lipschitz condition implies a lower bound in a similar way that the upper Lipschitz condition usually implies an upper bound. Our particular scenario is interesting because the method we use to prove the dimension bounds (the lower one in particular) differs from the “standard” technique for similar settings.

We will define a (probabilistic) iterated function system (IFS) as a set $\mathbb{X} \subset \mathbb{R}^d$ associated with a family of maps $\mathcal{W} = \{w_i\}_{i \in M}$, $w_i : \mathbb{X} \rightarrow \mathbb{X}$, where the maps are chosen independently according to a probability vector $\mathbf{p} = \{p_i\}_{i \in M}$, where $p_i > 0$ for all $i \in M$ and $\sum_{i \in M} p_i = 1$. The index set M is either finite or (countably) infinite. We will assume that the maps are bi-Lipschitz: for every $i \in M$ there exist constants Γ_i and γ_i , such that $0 < \gamma_i < \Gamma_i$ and

$$|w_i(x) - w_i(y)| \geq \gamma_i |x - y| \quad (1.1)$$

$$|w_i(x) - w_i(y)| \leq \Gamma_i |x - y|, \quad (1.2)$$

for all $x, y \in \mathbb{X}$. We will use the notation $\{\mathbb{X}, \mathcal{W}, \mathbf{p}\}$ for an IFS as described above.

Let $M^\infty = M \times M \times \dots$ and define the infinite-fold product probability measure on M^∞ as $\mathbb{P} = \mathbf{p} \times \mathbf{p} \times \dots$. We define the mapping $\pi : M^\infty \rightarrow \mathbb{R}^d$ by

$$\pi(i_1, i_2, \dots) = \lim_{n \rightarrow \infty} w_{i_1} \circ \dots \circ w_{i_n}(x_0) \quad (1.3)$$

if the limit exists.

For an IFS satisfying (1.2), it is well known (see [3]) that if there exists some $x \in \mathbb{X}$ such that the conditions

$$(i) \quad \sum_{i \in M} p_i \Gamma_i < \infty \quad (1.4)$$

$$(ii) \quad \sum_{i \in M} p_i |w_i(x) - x| < \infty \quad (1.5)$$

$$(iii) \quad -\infty < \sum_{i \in M} p_i \log \Gamma_i < 0 \quad (1.6)$$

hold, then there exists a unique probability measure μ on \mathbb{X} such that

$$\mu = \mathbb{P} \circ \pi^{-1}$$

on $\mathcal{B}(\mathbb{X})$, the Borel algebra on \mathbb{X} . Alternatively, μ may be defined as the unique measure satisfying

$$\mu = \sum_{i \in M} p_i \mu \circ w_i^{-1}.$$

The operator $\sum_{i \in M} p_i \mu \circ w_i^{-1}$ is sometimes called the Markov operator, and μ is called the invariant measure of the IFS. An IFS satisfying conditions (1.4)-(1.6) is said to satisfy average contractivity. In this case there may be expanding maps in \mathcal{W} and thus π does not necessarily exist for all $\mathbf{i} \in M^\infty$. However, the limit (1.3) exists \mathbb{P} -a.e. and does not depend on x_0 .

Another common notion related to iterated function systems is the open set condition, which is satisfied if there exists an open set $\mathcal{O} \subset \mathbb{X}$ such that

$$(i) \quad w_i(\mathcal{O}) \subset \mathcal{O} \text{ for all } i \in M \quad (1.7)$$

$$(ii) \quad w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset \text{ for all } i \neq j. \quad (1.8)$$

If, in addition,

$$(iii) \quad \mu(\mathcal{O}) > 0 \quad (1.9)$$

we will say that the IFS satisfies the strong open set condition (SOSC). The SOSC is a means to limit the overlapping of the maps, which simplifies the geometry in ways essential to some of the results below.

The set of all finite or infinite iterated function systems consisting of average-contracting bi-Lipschitz maps that satisfy $\sum_{i \in M} p_i \log p_i > -\infty$ will be denoted Ξ . We now state the main result.

Theorem 1.1. *Let μ denote the invariant measure of $\{\mathbb{X}, \mathcal{W}, \mathbf{p}\} \in \Xi$. If the strong open set condition is satisfied, and $\sum_{i \in M} p_i \log \gamma_i > -\infty$, then the Hausdorff and packing dimensions of μ satisfy*

$$\begin{aligned} i) \quad & \underline{s} \leq \dim_H \mu \leq \dim_H^* \mu \leq \bar{s} \\ ii) \quad & \underline{s} \leq \dim_P \mu \leq \dim_P^* \mu \leq \bar{s} \end{aligned}$$

where

$$\underline{s} = \frac{\sum_M p_i \log p_i}{\sum_M p_i \log \gamma_i} \text{ and } \bar{s} = \frac{\sum_M p_i \log p_i}{\sum_M p_i \log \Gamma_i}.$$

2 Preliminaries

First, we introduce some notation related to the product space $M^\infty = M \times M \times \dots$, which consists of infinite sequences of elements from M . Here \mathbb{N} will denote the set $\{1, 2, 3, \dots\}$. For any $\mathbf{i} = \{i_1, i_2, \dots\} \in M^\infty$, define $\mathbf{i}|k = \{i_1, i_2, \dots, i_k\}$ and $\mathbf{i}|j, k = \{i_j, i_{j+1}, \dots, i_k\}$ for $j < k$. For any $k, m \in \mathbb{N}$ and any $\mathbf{i}|k \in M^k$, $\mathbf{j}|m \in M^m$ we will use the operation $\mathbf{i}|k \circ \mathbf{j}|m = i_1, \dots, i_k, j_1, \dots, j_m$. By the notation $w_{\mathbf{i}|n}(x)$ we refer to the composition $w_{i_1} \circ w_{i_2} \circ \dots \circ w_{i_n}(x)$. In sums and products we will abbreviate the indexing in the fashion $\sum_{i \in \mathbf{i}|n} p_i = \sum_{\mathbf{i}|n} p_i$.

We define the overlapping set as

$$\Theta = \{x \in \mathbb{X} : \exists \mathbf{i}, \mathbf{j} \in M^\infty \text{ such that } \mathbf{i} \neq \mathbf{j} \text{ and } x = \pi(\mathbf{i}) = \pi(\mathbf{j})\}.$$

Let $C_{\mathbf{j}}(n) = \{\mathbf{i} \in M^\infty : \mathbf{i}|n = \mathbf{j}|n\}$, where $n \in \mathbb{N}$ and $\mathbf{j} \in M^\infty$. A set of this type is called a cylinder set. Cylinder sets generate a natural topology on M^∞ and are clopen sets. For any $\mathbf{i} \in M^\infty$ and $A \subset M^\infty$, define

$$\delta_{\mathbf{i}}(A, n) = \text{card} \left(A \cap \{\sigma^k(\mathbf{i})\}_{k=0}^{n-1} \right),$$

where σ denotes the left-shift operator on M^∞ , ie. $\sigma(\{i_1, i_2, \dots\}) = \{i_2, i_3, \dots\}$. Let

$$\mathcal{S}_0 = \left\{ \mathbf{i} \in M^\infty : A \in \mathcal{B}(M^\infty) \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \delta_{\mathbf{i}}(A, n) = \mathbb{P}(A) \right\},$$

where $\mathcal{B}(M^\infty)$ is the Borel algebra on M^∞ (using the aforementioned cylinder topology). Applying Birkhoff's ergodic theorem to the probability space $\{M^\infty, \mathcal{B}(M^\infty), \mathbb{P}\}$ and the shift map σ , we have

$$\mathbb{P}(\mathcal{S}_0) = 1. \quad (2.1)$$

Now, for a sequence of i.i.d random variables X_i with expectancy $\mathbb{E}(X_i) < \infty$, let

$$\mathcal{N}(X_i) = \left\{ \mathbf{i} \in M : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_{i_k} = \mathbb{E}(X_i) \right\}.$$

We may view $X_i^1 = |w_i(x) - x|$, $X_i^2 = \log \Gamma_i$ and $X_i^3 = \log p_i$ as sequences of i.i.d random variables with distribution given by \mathbf{p} . In light of conditions (1.4)-(1.6), $\mathbb{E}(X_i^k)$ exists for $k = 1, 2, 3$ (independently of x , see [3], corollary 5.2). We now define

$$\mathcal{S} = \mathcal{S}_0 \cap \left(\bigcap_{k=1}^3 \mathcal{N}(X_i^k) \right).$$

By (2.1) and the strong law of large numbers, we have $\mathbb{P}(\mathcal{S}) = 1$. We will denote

$$\mathcal{A} = \pi(\mathcal{S}).$$

While known that π exists for almost every $\mathbf{i} \in M^\infty$, the following lemma shows specifically that π is defined everywhere on \mathcal{S} .

Lemma 2.1. *π is well-defined on \mathcal{S} .*

Proof. Choose $\mathbf{i} \in \mathcal{S}$ and $x \in \mathbb{X}$ arbitrarily. Denote $\mathbb{E}(|w_i(x) - x|) = \lambda_1$ and $\mathbb{E}(\log \Gamma_i) = \lambda_2$. Fix $c_1 > \lambda_1$, $\lambda_2 < c_2 < 0$ and let

$$K = c_1 \sum_{k=0}^{\infty} (k+2) e^{kc_2}.$$

Fix $\epsilon > 0$ such that $\epsilon < \min \{c_i - \lambda_i\}_{i=1}^2$. We can choose $N \geq 1$ large enough that the conditions

$$|w_{i_n}(x) - x| \leq \sum_{k=1}^n |w_{i_k}(x) - x| \leq n(\lambda_1 + \epsilon) \quad (2.2)$$

$$\prod_{k=1}^n \Gamma_{i_k} = e^{\sum_{k=1}^n \log \Gamma_{i_k}} \leq e^{n(\lambda_2 + \epsilon)} \quad (2.3)$$

$$ne^{nc_2} K < \epsilon \quad (2.4)$$

all hold for all $n \geq N$. Note that, by the triangle inequality, we have

$$\begin{aligned} |w_{\mathbf{i}|n,m}(x) - x| &\leq |w_{i_n}(x) - x| + \Gamma_{i_n} |w_{i_{n+1}}(x) - x| + \\ &\quad \dots + \Gamma_{i_n} \dots \Gamma_{i_{m-1}} |w_{i_m}(x) - x| \end{aligned}$$

for any $\mathbf{i} \in M^\infty$ and $m \geq n$. Now consider, for $m \geq n > N$,

$$\begin{aligned} |w_{\mathbf{i}|m}(x) - w_{\mathbf{i}|n}(x)| &\leq \prod_{k=1}^n \Gamma_{i_k} |w_{\mathbf{i}|n+1,m}(x) - x| \\ &\leq \sum_{j=1}^{m-n} \left(\prod_{k=1}^{n+j-1} \Gamma_{i_k} |w_{i_{n+j}}(x) - x| \right) \\ &\leq \sum_{j=1}^{m-n} e^{(n+j-1)(\lambda_2 + \epsilon)} (n+j)(\lambda_1 + \epsilon) \\ &< e^{nc_2} \cdot c_1 \sum_{j=1}^{\infty} e^{(j-1)c_2} (n+j) \\ &< ne^{nc_2} K \\ &< \epsilon. \end{aligned}$$

Thus, the sequence $w_{\mathbf{i}|n}(x)$ is Cauchy, meaning that $\pi(\mathbf{i}) = \lim_{n \rightarrow \infty} w_{\mathbf{i}|n}(x)$ exists. Since x was chosen arbitrarily, the convergence is independent of x . \square

Observe that, for any integers $n \geq m$ and $\mathbf{i}, \mathbf{j} \in M^\infty$, $\delta_{\mathbf{i}}(C_{\mathbf{j}}(m), n)$ denotes the number of times the sequence $\mathbf{j}|m$ appears in the sequence $\mathbf{i}|m + n - 1$. Consequently, for $\mathbf{i} \in \mathcal{S}$ and $\mathbf{j} \in M^\infty$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \delta_{\mathbf{i}}(C_{\mathbf{j}}(m), n) = \mathbb{P}(C_{\mathbf{j}}(m)) = \prod_{\mathbf{j}|m} p_i. \quad (2.5)$$

For $\mathbf{i} \in \mathcal{S}$ and $n \in \mathbb{N}$, define

$$\tau_{\mathbf{i}}(A, n) = \min \{t : \sigma^t(\mathbf{i}) \in A, \delta_{\mathbf{i}}(A, t) = n - 1\}.$$

The above expression is well-defined for every \mathbb{P} -positive $A \in \mathcal{B}(M^\infty)$.

Note that, for any $\mathbf{i} \in M^\infty$, $n \in \mathbb{N}$ and $E \subset \mathbb{X}$,

$$\mu(w_{\mathbf{i}|n}(E)) \geq \mathbb{P}\{\mathbf{j} : \mathbf{j}|n = \mathbf{i}|n, \pi(j_{n+1}, j_{n+2}, \dots) \in E\} = \prod_{\mathbf{i}|n} p_i \mu(E). \quad (2.6)$$

The inequality in the above equation arises from the fact that we may have points in $w_{\mathbf{i}|n}(E)$ not necessarily of the form $\pi(i_1, \dots, i_n, \dots)$, since there may be several sequences in M^∞ corresponding to the same point in \mathbb{X} . However, in lemma 3.1 it is shown that π is injective on \mathcal{S} if the SOSC holds.

We now give some notation and definitions related to fractal geometry. For any $\delta > 0$, a countable (or finite) collection of sets $\{U_i\}$ is a δ -cover of a set E if $E \subset \bigcup_i U_i$ and $\text{diam } U_i \leq \delta$ for all i . Similarly, a countable collection of pairwise disjoint balls $\{B_i\}$, with centres in E and radii at most δ , is called a δ -packing of E .

For $\delta > 0$ and $E \subset \mathbb{R}^d$ the s -dimensional Hausdorff measure \mathcal{H}^s is defined by

$$\mathcal{H}^s(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

The s -dimensional packing pre-measure \mathcal{P}_0^s is given by

$$\mathcal{P}_0^s(E) = \limsup_{\delta \rightarrow 0} \left\{ \sum_i |B_i|^s : \{B_i\} \text{ is a } \delta\text{-packing of } E \right\},$$

whereby the s -dimensional packing measure \mathcal{P}^s is defined by

$$\mathcal{P}^s(E) = \inf \left\{ \sum_i \mathcal{P}_0^s(E_i) : E \subset \bigcup_i E_i \right\}.$$

Now we can define the Hausdorff and packing dimensions of E by

$$\begin{aligned} \dim_H(E) &= \sup \{s : \mathcal{H}^s(E) = \infty\} = \inf \{s : \mathcal{H}^s(E) = 0\} \\ \dim_P(E) &= \sup \{s : \mathcal{P}^s(E) = \infty\} = \inf \{s : \mathcal{P}^s(E) = 0\} \end{aligned}$$

We will use two common definitions of the dimension of a measure. The lower and upper Hausdorff dimensions of a probability measure μ are given by $\dim_H \mu = \inf \{\dim_H(E) : \mu(E) > 0\}$ and $\dim_H^* \mu = \inf \{\dim_H(E) : \mu(E) = 1\}$, respectively. Similarly, we define the upper and lower packing dimensions of μ to be $\dim_P \mu = \inf \{\dim_P(E) : \mu(E) > 0\}$ and $\dim_P^* \mu = \inf \{\dim_P(E) : \mu(E) = 1\}$. The upper and lower local dimensions of μ at $x \in \mathbb{R}^d$ are given by

$$\underline{\dim}_{\text{loc}} \mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \text{ and } \overline{\dim}_{\text{loc}} \mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$

where $B(x, r)$ is the closed ball centered around x with radius r . As seen in [2], the above notions of dimension may be equivalently defined by

$$\dim_H \mu = \sup \{s : \underline{\dim}_{\text{loc}} \mu(x) \geq s \text{ for } \mu\text{-almost all } x\} \quad (2.7)$$

$$\dim_H^* \mu = \inf \{s : \underline{\dim}_{\text{loc}} \mu(x) \leq s \text{ for } \mu\text{-almost all } x\}, \quad (2.8)$$

and

$$\dim_P \mu = \sup \{s : \overline{\dim}_{\text{loc}} \mu(x) \geq s \text{ for } \mu\text{-almost all } x\} \quad (2.9)$$

$$\dim_P^* \mu = \inf \{s : \overline{\dim}_{\text{loc}} \mu(x) \leq s \text{ for } \mu\text{-almost all } x\}. \quad (2.10)$$

3 Results

The following is a generalization of theorem 2.1 in [7], and is a key step in proving the lower bound in theorem 1.1.

Lemma 3.1. *Let Θ be the overlapping set for an IFS $\{\mathbb{X}, \mathcal{W}, \mathbf{p}\} \in \Xi$ which also satisfies the SOSC. Then*

$$\mu(\Theta) = 0.$$

Proof. Let $\pi_n(\mathbf{i}) = \pi \circ \sigma^{n-1}(\mathbf{i})$ and define the orbit of $\mathbf{i} \in \mathcal{S}$ by

$$O(\mathbf{i}) = \{\pi_n(\mathbf{i})\}_{n=1}^\infty.$$

By lemma 2.1, the above orbit is well-defined. Also, note that the symbolic orbit $\{\sigma^n(\mathbf{i})\}_{n=0}^\infty$ is dense in M^∞ .

Suppose there exist $\mathbf{i}, \mathbf{j} \in \mathcal{S}$ such that $\pi(\mathbf{i}) = \pi(\mathbf{j})$ and $\mathbf{i} \neq \mathbf{j}$. Let $k = \min\{n : i_n \neq j_n\}$, ie. k is the first position where \mathbf{i} and \mathbf{j} differ. By (1.1) the maps $\{w_i\}_{i \in M}$ are injective, so $\pi_k(\mathbf{i}) = \pi_k(\mathbf{j})$. Furthermore, $\pi_k(\mathbf{i}) \in w_{i_k}(\mathcal{A})$ and $\pi_k(\mathbf{j}) \in w_{j_k}(\mathcal{A})$, so by (1.8), at least one of $\pi_k(\mathbf{i}) \in w_{i_k}(\mathcal{A} \setminus \mathcal{O})$ and $\pi_k(\mathbf{j}) \in w_{j_k}(\mathcal{A} \setminus \mathcal{O})$ must hold. Assuming $\pi_k(\mathbf{i}) \in w_{i_k}(\mathcal{A} \setminus \mathcal{O})$, we have $\pi_{k+1}(\mathbf{i}) \in \mathcal{A} \setminus \mathcal{O}$, from which it follows that $\pi_{k+m}(\mathbf{i}) \in \mathcal{A} \setminus \mathcal{O}$ for every $m \geq 1$, since if $\pi_{k+m}(\mathbf{i}) \in \mathcal{O}$ for some $m \geq 1$, then by (1.7) we have $\pi_{k+1}(\mathbf{i}) \in \mathcal{O}$, which is a contradiction.

Now, let

$$E = \{x \in \mathbb{X} : \mu(B(x, r)) > 0 \ \forall r > 0\}$$

and note that $\mu(\mathcal{A} \cap E) = 1$ since its complement is clearly a null set. By definition of \mathcal{S} , for every $x \in \mathcal{A} \cap E$ and $\delta > 0$ we can find a $k > 0$ such that $\pi_k(\mathbf{i}) \in B(x, \delta)$, since $\mathbb{P}(\pi^{-1}(B(x, \delta))) > 0$. Thus the orbit $O(\mathbf{i}) \subset \mathcal{A} \setminus \mathcal{O}$ is dense in $\mathcal{A} \cap E$, which implies that $(\mathcal{A} \cap E) \cap \mathcal{O} = \emptyset$ and consequently that $\mu(\mathcal{O}) = 0$, contradicting (1.9). Thus $\mathcal{A} \cap \Theta = \emptyset$ and the proposition follows. \square

An immediate corollary is that if the SOSC is satisfied, then for any $\mathbf{i} \in M^\infty$ and $n \in \mathbb{N}$ we have

$$\mu(w_{\mathbf{i}|n}(\mathcal{A})) = \mu(w_{\mathbf{i}|n}(\overline{\mathcal{O}})) = \prod_{i|n} p_i. \quad (3.1)$$

Lemma 3.2. *Let $\mathbf{i} \in \mathcal{S}$. For any $A \in \mathcal{B}(M^\infty)$ with $\mathbb{P}(A) > 0$ we have*

$$\lim_{n \rightarrow \infty} \frac{\tau_{\mathbf{i}}(A, n)}{\tau_{\mathbf{i}}(A, n-1)} = 1.$$

Proof. Fix $\mathbf{i} \in \mathcal{S}$ and $A \in \mathcal{B}(M^\infty)$ (with $\mathbb{P}(A) > 0$). The result follows immediately upon observing that

$$\mathbb{P}(A) = \lim_{n \rightarrow \infty} \frac{\delta_{\mathbf{i}}(A, n)}{n} = \lim_{k \rightarrow \infty} \frac{k}{\tau_{\mathbf{i}}(A, k)}.$$

□

The proof of theorem 1.1 is now given by the following two lemmas.

Lemma 3.3. *Let μ be the invariant measure of $\{\mathbb{X}, \mathcal{W}, \mathbf{p}\} \in \Xi$. Then*

$$\dim_P^* \mu \leq \frac{\sum_i p_i \log p_i}{\sum_i p_i \log \Gamma_i}.$$

Proof. First note that, for any $\mathbf{i} \in \mathcal{S}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{\mathbf{i}|n} \Gamma_i = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\mathbf{i}|n} \log \Gamma_i = \sum_M p_i \log \Gamma_i,$$

Thus

$$\lim_{n \rightarrow \infty} \prod_{\mathbf{i}|n} \Gamma_i = 0, \tag{3.2}$$

by condition (1.6). Now fix $x_0 \in \mathbb{X}$, $\epsilon > 0$ and choose R_ϵ such that $\mu(B(x_0, R_\epsilon)) > 1 - \epsilon$. Denote the ball $B(x_0, R_\epsilon)$ by B , and let $B' = \pi^{-1}(B)$. Since π is measurable, B' is Borel, something which can also be seen by writing

$$\pi^{-1}(B) = \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ \mathbf{i} \in M^\infty : \inf_{x \in B} d(w_{i_1} \circ \cdots \circ w_{i_n}(x_0), x) < \frac{1}{K} \right\},$$

where $x_0 \in \mathbb{X}$ is arbitrary. The set within the brackets above is open (since it is a countable union of cylinder sets) implying $\pi^{-1}(B) \in \mathcal{B}(M^\infty)$. Now, choose $x = \pi(\mathbf{i}) \in B \cap \mathcal{A}$ and fix $r > 0$. Let

$$N = \min \left\{ k : \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', k)} \Gamma_i < \frac{r}{2R_\epsilon} \leq \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', k-1)} \Gamma_i \right\}.$$

By (3.2), the above constant exists, provided that r is chosen small enough. Note that, by the definition of $\tau_{\mathbf{i}}(B', k)$, we have $x \in w_{\mathbf{i}|\tau_{\mathbf{i}}(B', k)}(B)$ for all $k \geq 1$. Since $|w_{\mathbf{i}|n}(B)| \leq \prod_{\mathbf{i}|n} \Gamma_i 2R_\epsilon$ (here $|\cdot|$ denotes the diameter of a set), the set $w_{\mathbf{i}|\tau_{\mathbf{i}}(B', N)}(B)$ is certainly included in the ball $B(x, r)$. Combining this with (2.6) yields

$$\begin{aligned} \frac{\log \mu(B(x, r))}{\log r} &\leq \frac{\log \mu(w_{\mathbf{i}|\tau_{\mathbf{i}}(B', N)}(B))}{\log \left(2R_\epsilon \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', N-1)} \Gamma_i \right)} \\ &\leq \frac{\log \mu(B) + \log \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', N)} p_i}{\log 2R_\epsilon + \log \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', N-1)} \Gamma_i}. \end{aligned}$$

Applying lemma 3.2, we get

$$\begin{aligned}
 \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} &\leq \\
 \limsup_{N \rightarrow \infty} \frac{\tau_{\mathbf{i}}(B', N)}{\tau_{\mathbf{i}}(B', N-1)} &\cdot \frac{\frac{1}{\tau_{\mathbf{i}}(B', N)} \left(\log \mu(B) + \log \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', N)} p_i \right)}{\frac{1}{\tau_{\mathbf{i}}(B', N-1)} \left(\log 2R_\epsilon + \log \prod_{\mathbf{i}|\tau_{\mathbf{i}}(B', N-1)} \Gamma_i \right)} \\
 &= \frac{\sum_M p_i \log p_i}{\sum_M p_i \log \Gamma_i}
 \end{aligned}$$

from which we acquire $\overline{\dim}_{\text{loc}} \mu(x) \leq \bar{s}$ for all $x \in \mathcal{A} \cap B$. Now let

$$E_n = B(x_0, R_{\epsilon/n}) \cap \mathcal{A},$$

where $\{R_{\epsilon/n}\}_{n=1}^\infty$ is chosen to be an increasing sequence such that

$$\mu(B(x_0, R_{\epsilon/n})) > 1 - \frac{\epsilon}{n}.$$

For every $n \geq 1$, the above argument can be repeated so that $\overline{\dim}_{\text{loc}} \mu(x) \leq \underline{s}$ for all $x \in E_n$. Since $\mu\left(\bigcup_{n \geq 1} E_n\right) = \mu(\mathcal{A}) = 1$, the result follows from (2.10). \square

Remark. Note that the SOSOC need not be assumed in the above lemma.

Lemma 3.4. *Let μ be the invariant measure of $\{\mathbb{X}, \mathcal{W}, \mathbf{p}\} \in \Xi$ and assume that the SOSOC holds. If $\sum_{i \in M} p_i \log \gamma_i > -\infty$, then*

$$\dim_H \mu \geq \frac{\sum_i p_i \log p_i}{\sum_i p_i \log \gamma_i}.$$

Proof. Fix $\epsilon > 0$ and $x_0 \in \mathbb{X}$. As before, we choose R_ϵ such that $\mu(B(x_0, R_\epsilon)) > 1 - \epsilon$, and denote the ball $B(x_0, R_\epsilon)$ by B . Furthermore, choose some $\mathbf{j} \in \mathcal{S}$ and integer K such that $\pi(\mathbf{j}) \in \mathcal{O}$ and $d(w_{\mathbf{j}|K}(B), \partial\mathcal{O}) > \delta$ for some sufficiently small $\delta > 0$. Moreover, choose $x = \pi(\mathbf{i}) \in B \cap \mathcal{A}$ and note that \mathbf{i} is unique (due to lemma 3.1) given this x . Fix $r > 0$ and let

$$F = C_{\mathbf{j}}(K) \cap \{\mathbf{i} \in \mathcal{S} : \pi(\sigma^K(\mathbf{i})) \in B\}.$$

Now define

$$N = \max \left\{ k : \prod_{\mathbf{i}|\tau_{\mathbf{i}}(F, k)} \gamma_i < \frac{r}{\delta} \leq \prod_{\mathbf{i}|\tau_{\mathbf{i}}(F, k-1)} \gamma_i \right\}.$$

As before, N exists provided that r is small enough. For any n , the map $w_{\mathbf{i}|n}$

satisfies (1.1) with lower Lipschitz constant $\prod_{i|n} \gamma_i$, so

$$\begin{aligned}
 B(x, r) &\subset B\left(x, \delta \prod_{i|\tau_1(F, N-1)} \gamma_i\right) \\
 &\subseteq w_{i|\tau_1(F, N-1)} \left[B\left(\pi \circ \sigma^{\tau_1(F, N-1)}(\mathbf{i}), \delta\right) \right] \\
 &= w_{i|\tau_1(F, N-1)} \left[B\left(w_{j|K} \circ \pi \circ \sigma^{\tau_1(F, N-1)+K}(\mathbf{i}), \delta\right) \right] \\
 &\subset w_{i|\tau_1(F, N-1)}(\mathcal{O}) \\
 &\subset w_{i|\tau_1(F, N-1)}(\overline{\mathcal{O}}),
 \end{aligned}$$

since, by definition, we have that $B(w_{j|K}(y), \delta) \subset \mathcal{O}$ for any $y \in B$. Consequently,

$$\mu(B(x, r)) \leq \mu(w_{i|\tau_1(F, N-1)}(\overline{\mathcal{O}})) = \prod_{i|\tau_1(F, N-1)} p_i,$$

giving

$$\frac{\log \mu(B(x, r))}{\log r} \geq \frac{\log \prod_{i|\tau_1(F, N-1)} p_i}{\log \delta \prod_{i|\tau_1(F, N)} \gamma_i}.$$

We now arrive at

$$\begin{aligned}
 \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} &\geq \\
 \liminf_{N \rightarrow \infty} \frac{\tau_1(F, N-1)}{\tau_1(F, N)} &\cdot \frac{\frac{1}{\tau_1(F, N-1)} \log \prod_{i|\tau_1(F, N-1)} p_i}{\frac{1}{\tau_1(F, N)} (\log \delta + \log \prod_{i|\tau_1(F, N)} \gamma_i)} \\
 &= \frac{\sum_M p_i \log p_i}{\sum_M p_i \log \gamma_i}
 \end{aligned}$$

Again, we applied lemma 3.2 since $\mathbb{P}(F) = \prod_{j|K} p_i \cdot \mu(B) > 0$. We have now shown that $\underline{\dim}_{\text{loc}} \mu(x) \geq \underline{s}$ for all $x \in B \cap \mathcal{A}$. Similarly to lemma 3.3, let $E_n = B(x_0, R_{\epsilon/n}) \cap \mathcal{A}$, and the result follows by (2.7). \square

We may also remark that by standard results (see eg. [7], theorem 4.4 in [6] and proposition 2.2 in [2]) the following result holds.

Corollary 3.5. *Under the conditions in theorem 1.1 we have that*

- i) μ is absolutely continuous w.r.t. \mathcal{H}^d and \mathcal{P}^d whenever $d < \underline{s}$
- ii) μ is singular w.r.t. \mathcal{H}^d and \mathcal{P}^d whenever $d > \bar{s}$

4 Examples in \mathbb{R}^2

The following examples demonstrate cases where all the conditions of Theorem 1.1 hold. The maps aren't all strictly contracting, and the SOSC holds, but the fine structure of μ can still be argued to be "fractal". The invariant measures are visualized by plotting the points of the Markov chain (the so-called forward iteration)

$$w_{i_1}(x_0), w_{i_2} \circ w_{i_1}(x_0), \dots, w_{i_N} \circ \dots \circ w_{i_1}(x_0)$$

for some starting point x_0 , where at each step i_n is chosen (i.i.d) according to the given probability vector.

Example 4.1. Consider an IFS on $\mathbb{X} = [0, \infty) \times [0, 1]$ given by

$$w_i(x, y) = \begin{pmatrix} \Gamma_i & 0 \\ 0 & \gamma_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} i-1 \\ \sum_{k=1}^{i-1} \gamma_k \end{pmatrix}$$

where

$$\gamma_i = \frac{2^{i-1}}{3^i}, \quad \Gamma_i = \gamma_i + \frac{i}{10}$$

for $i = 1, 2, \dots$. We define the probabilities by $p_i = 2^{-i}$.

Since the maps are affine it is not hard to see that Γ_i and γ_i indeed are the upper and lower Lipschitz constants for w_i , respectively. In this particular case the maps are expanding horizontally and contracting vertically. The maps split \mathbb{X} into smaller and smaller slices, increasingly displaced from the origin horizontally (see figure 4.1a). It is also apparent from the same figure that the SOSC is satisfied with $\mathcal{O} = (0, \infty) \times (0, 1)$.

Furthermore, we have $\Gamma_i > 1$ for all $i \geq 10$, and $\gamma_i < 1$ for all $i \geq 1$. The IFS belongs to Ξ since $\sum_{i=1}^{\infty} p_i \log p_i \approx -1.39$ and

$$\sum_{i=1}^{\infty} p_i \Gamma_i \approx 0.45, \quad \sum_{i=1}^{\infty} p_i \log \Gamma_i \approx -0.805, \quad \sum_{i=1}^{\infty} p_i |w_i((0, 0))| \approx 1.03,$$

whereby the conditions (1.4)-(1.6) are all satisfied. By theorem 1.1 we now have

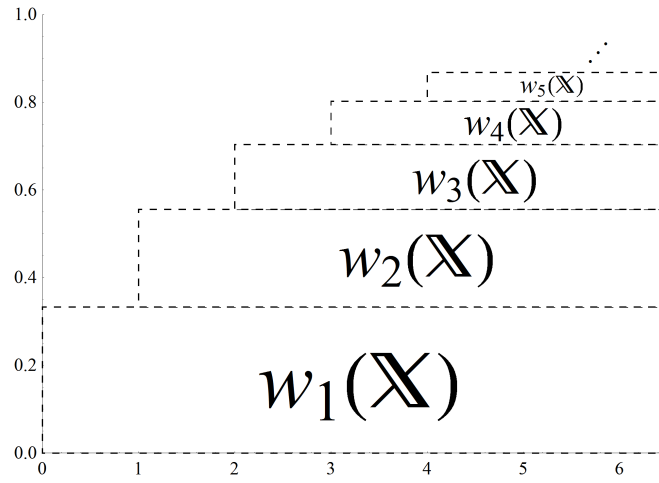
$$0.92 < \dim_H \mu \leq \dim_P^* \mu < 1.73.$$

The invariant measure is visualized in figure 4.1b.

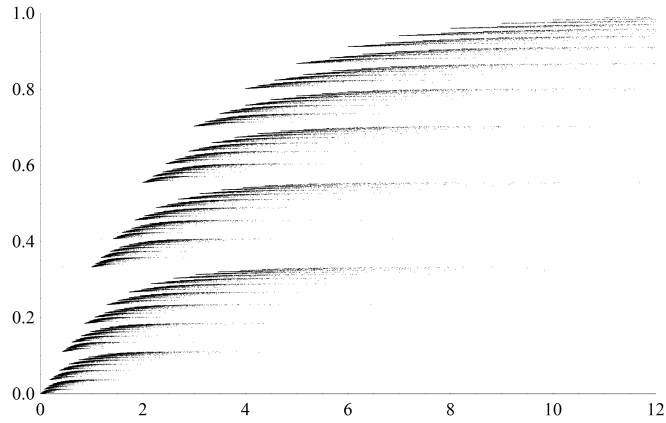
Example 4.2. Define an IFS on the two-dimensional unit ball C , consisting of four maps, by $w_i(x, y) = a_i \begin{pmatrix} x \\ y \end{pmatrix} + b_i$, where

$$a_i = \begin{cases} \frac{1}{4} \left(5 - 3(x^2 + y^2)^{1/6} \right), & i = 1, 2 \\ \frac{1}{3}, & i = 3, 4 \end{cases}$$

Figure 4.1: Example 4.1

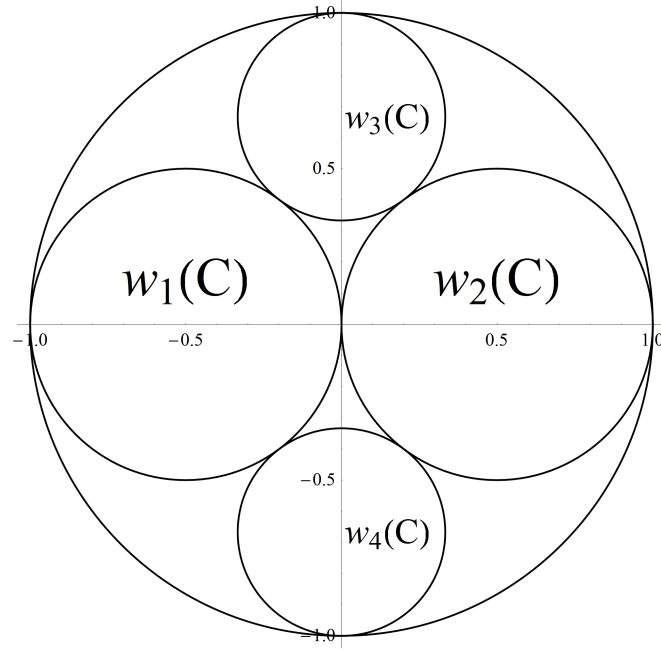


(a) Schematic overview of the maps

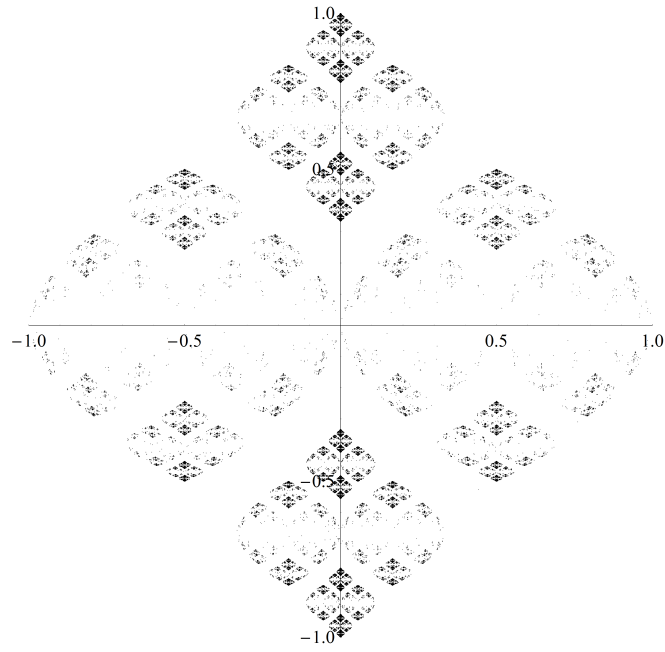


(b) Plot of the Markov chain

Figure 4.2: Example 4.2



(a) Schematic overview of the maps



(b) Plot of the Markov chain

and

$$b_1 = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ -2/3 \end{pmatrix}.$$

Let $\mathbf{p} = \{0.1, 0.1, 0.4, 0.4\}$. This IFS maps C into four smaller circles (see the schematic overview in figure 4.2). The maps w_3 and w_4 are similitudes (ie. $\Gamma = \gamma = 1/3$) while w_1 and w_2 expand locally, around the circle centres, but contract toward the edges. Thus w_1 and w_2 are not strict contractions, but the support of the invariant measure μ is still bounded and contained in C . It can be shown that $\Gamma_i = 5/4$ and $\gamma_i = 1/4$ for $i = 1, 2$.

The SOSC is satisfied (with \mathcal{O} being the open unit ball), and we have $\sum p_i \log \Gamma_i \approx -0.74$ so average contractivity is fulfilled as well. By theorem 1.1,

$$1.05 < \dim_H \mu \leq \dim_P^* \mu < 1.43.$$

Note that the dimension is strictly smaller than that of the Sierpinski gasket ($\log 3 / \log 2 \approx 1.58$) which would have been achieved if all the maps were similitudes.

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